Computing the Swap Rate

Swap Rate

Suppose that a swap determines the rate at times:

$$T_0, \dots, T_{N-1}$$

and pays at times:

$$T_1, \dots, T_N.$$

The fixed side of the swap will pay:

$$s(t) \sum_{i=0}^{N-1} (T_{i+1} - T_i) P(t, T_{i+1})$$

where $s(t)$ is called the swap rate and it moves as a function of t as the yield curve moves.

Swap Rate

How much is the floating side of the swap worth?

If $T_0 = 0$ then the floating side of the swap would be very similar to a floating coupon-paying bond.

The only difference being the principal payment at the end.

Then, the value woud be:

$$1 - P(t, T_N)$$

However, since T_0 might not be zero what we need to do is to subtract all the cash-flows between time 0 and T_0. But these cash-flows are also very similar to the value of a floating coupon-paying bond. They are worth:

$$1 - P(t, T_0)$$

Swap Rate

The value of the floating side of the swap is then:

$$P(t, T_0) - P(t, T_N)$$

Equating both sides:

$$s(t) \sum_{i=0}^{N-1} (T_{i+1} - T_i) P(t, T_{i+1}) = P(t, T_0) - P(t, T_N)$$

which gives that:

$$s(t) = \frac{P(t, T_0) - P(t, T_N)}{\sum_{i=0}^{N-1} (T_{i+1} - T_i) P(t, T_{i+1})}$$

Swap Rate

Another way to generalizing to forward starting swaps:

We know that, for a swap starting immediately, the swap rate satisfies:

$$s(t)\sum_{i=0}^{N-1}(T_{i+1}-T_i)P(t,T_{i+1}) = 1 - P(t,T_N)$$

Let us we now consider a swap starting at a future time T_0.

Swap Rate

If market values were not to change between now and time T_0 the swap rate, at the beginning of the swap would solve:

$$\hat{s}(T_0)\sum_{i=0}^{N-1}(T_{i+1}-T_i)\hat{P}(T_0,T_{i+1}-T_0)=1-\hat{P}(T_0,T_N-T_0)$$

where $\hat{P}$ denotes the fact that those are the prices of the zero-coupon bonds IF the zero rates end up being the ones implied by the curve (the forwards) today.

Swap Rate

Multiplying both sides by $P(t, T_0)$ we get:

$$LHS = \hat{s}(T_0) \sum_{i=0}^{N-1} (T_{i+1} - T_i) \hat{P}(T_0, T_{i+1} - T_0)\, P(t, T_0)$$

$$RHS = P(t, T_0) - \hat{P}(T_0, T_N - T_0)\, P(t, T_0)$$

Now, $\hat{P}(T_0, T_i - T_0)\, P(t, T_0)$ is just $P(t, T_i)$ and, therefore:

$$\hat{s}(T_0) \sum_{i=0}^{N-1} (T_{i+1} - T_i) P(t, T_{i+1}) = P(t, T_0) - P(t, T_N)$$

Swap Rate

Which indicates that $\hat{s}(T_0)$ has to be equal to $s(t)$.

This says that, in order to compute the swap rate in a forward started swap, we can just compute it as a regular swap assuming that we are now at time T_0 and that the forwards are the realized zero-coupon rates.