Stochastic Integration

Remember...

If F is a continuously differentiable function with:

$$f(s) = \frac{dF(s)}{ds}$$

then:

$$\int_0^t X(s)dF(s) = \int_0^t X(s)f(s)ds$$

If F is not differentiable then we can still compute the integral but in Riemann(or Lebesgue)-Stieltjes sense:

$$\int_0^t X(s)dF(s) = \lim_{n\to\infty} \sum_{k=1}^{n} X(\frac{(k-1)t}{n})(F(\frac{kt}{n}) - F(\frac{(k-1)t}{n}))$$

As we mentioned already...

We can't do the same thing if we want to replace F by W.

So we need to construct an integral in a different sense.

A priori is not clear that it can be done..

Idea:

- See if we can make sense of an integration for a small set of integrands.
- Try to reach more integrands by taking some limit.

Simple Processes

A stochastic process $\{X_t\}_{t\in[0,T]}$ is called simple if there exist real numbers:

$$0 = t_0 < t_1 < ... < t_p = T$$

and bounded random variables $\Phi_i : \Omega \longrightarrow \mathbb{R}$ $(i = 1, ..., p)$ with

$$\Phi_0 \;\; \mathcal{F}_0\text{-measurable and } \Phi_i \;\; \mathcal{F}_{t_i-1}\text{-measurable}$$

such that:

$$X_t(\omega) = \Phi_0(\omega)1_0(t) + \sum_{i=1}^{p} \Phi_i(\omega)1_{(t_{i-1},t_i]}(t)$$

for each $\omega \in \Omega$.

Itô Integral for Simple Processes

In this case the integral can be defined in the usual way:

$$I_t(X) = \int_0^t X_s dW_s = \sum_{1 \le i \le k} \Phi_i (W_{t_i} - W_{t_{i-1}}) + \Phi_{k+1}(W_t - W_{t_k})$$

or, in a more compact notation:

$$I_t(X) = \sum_{1 \le i \le p} \Phi_i (W_{t_i \wedge t} - W_{t_{i-1} \wedge t})$$

where $\wedge$ denotes the minimum of the two values.

Properties of the Integral

If X is a simple process:

1. $I_t(X)$ is a continuous martingale.
   In particular $E(I_t(X)) = 0$ for all $t \in [0, T]$.

2. $E(I_t^2(X)) = E(\int_0^t X_s^2 ds)$.

Sketch of Proof: Part 1

We need to prove that:

$$E(I_t(X)|\mathcal{F}_s = I_s(X)$$

but

$$E(I_t(X)|\mathcal{F}_s = E(\sum_{1\leq i\leq k} \Phi_i(W_{t_i} - W_{t_{i-1}}) + \Phi_{k+1}(W_t - W_{t_k})|\mathcal{F}_s)$$

Suppose that i_0 is the index so that $i_0 < s \leq i_0 + 1$.

Therefore we can separate the sums into two groups, the terms that involve things prior to s and the term that involve things after s.

The terms that involve things prior to s are all measurable w.r.t $\mathcal{F}_s$ (and they amount to $I_s(X)$) and for the terms that involve things after s:

Sketch of Proof: Part 1

$$E(\Phi_i(W_{t_i} - W_{t_{i-1}})|\mathcal{F}_s) = 0 \ , \ t_{i-1} \geq s$$

Why? We can prove it using iterated expectations.

Notice that Φ_i is measurable wrt to $\mathcal{F}_{t_{i-1}}$ so, it is known at time t_{i-1}. Therefore, whatever Φ_i ends up being at time t_{i-1} it gets multiplied by a normal variable, so the expected value has to be zero.

Sketch of Proof: Part 2

Now, we have to deal with $E(I_t(X)^2)$

Distributing the expression for the integral we get a bunch of terms like:

$$E(\Phi_i \Phi_j (W_{t_i} - W_{t_{i-1}})(W_{t_j} - W_{t_{j-1}}))$$

Case $i \neq j$:
If $i > j$ then we can condition wrt $\mathcal{F}_{i-1}$:

$$E(E(\Phi_i \Phi_j (W_{t_i} - W_{t_{i-1}})(W_{t_j} - W_{t_{j-1}}))|\mathcal{F}_{i-1})$$

and observe that all the terms but the last one are measurables wrt $\mathcal{F}_{i-1}$ (they are all known before that) so:

$$E(\Phi_i \Phi_j (W_{t_j} - W_{t_{j-1}}) E((W_{t_i} - W_{t_{i-1}})|\mathcal{F}_{i-1})) = 0$$

Sketch of Proof: Part 2

Case $i = j$:

$$E(\Phi_i^2(W_{t_i} - W_{t_{i-1}})^2) = E(\Phi_i^2 E((W_{t_i} - W_{t_{i-1}})^2 | \mathcal{F}_{t_{i-1}}))$$

$$= E(\Phi_i^2(t_i - t_{i-1}))$$

So, the cross terms vanish and the $i = j$ terms are the only ones that survive, then:

$$E(I_t(X)^2) = E(\sum_{i=1}^{k} \Phi_i^2(t_i - t_{i-1})) = E(\int_0^t X_s^2 ds)$$

Measurability

A stochastic process X_t is **measurable** if the mapping:

$$[0, \infty) \longrightarrow \mathbb{R}^n$$

$$(s, \omega) \longrightarrow X_s(\omega)$$

is measurable in the product space.

We will need a related (but different) notion: A stochastic process X_t is **progressively measurable** if, for every t, the mapping:

$$[0, t) \longrightarrow \mathbb{R}^n$$

$$(s, \omega) \longrightarrow X_s(\omega)$$

is measurable (w.r.t the information up to time t).

Approximation

Theorem: IF X_t is a progressively measurable process so that $E(\int_0^T X_t^2 dt) < \infty$ then X_t can b approximated by simple processes $X^{(n)}$ so that:

$$\lim_{n\to\infty} E(\int_0^T (X_s - X_s^{(n)})^2 ds = 0$$

Construction of the Integral

How do we define an integral for general integrands?

If we call $L^2[0, T]$ the space of progressively measurable processes with $E(\int_0^T X_t^2 dt) < \infty$ then there exists a unique mapping J from $L^2[0, T]$ into the space of continuous martingales on $[0, T]$ so that:

1. IF X is a simple process then the mapping J coincides with the integral we defined before (with probability 1).
2. $E(J_t(X)^2) = E(\int_0^t X_s^2 ds)$.

This equality is called the Itô isometry.

We define J as the stochastic integral of X w.r.t. W and denote:

$$\int_0^t X_s dW_s = J_t(X)$$

Itô Formula.

Suppose that W_t is a Brownian motion and X_t:

$$X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s$$

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be twice continuously differentiable. Then:

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) H_s^2 ds$$

$$= f(X_0) + \int_0^t (f'(X_s) K_s + \frac{1}{2} \int_0^t f''(X_s) H_s^2) ds$$

$$+ \int_0^t f'(X_s) H_s dW_s$$

Itô Formula. Idea of proof

Let us consider a partition of $[0, t]$. By Taylor:

$$f(X_t) - f(X_0) = \sum_{k=1}^{m} (f(X_{t_k}) - f(X_{t_{k-1}}))$$

$$= \sum_{k=1}^{m} f'(X_{t_{k-1}})(X_{t_k} - X_{t_{k-1}})$$

$$+ \frac{1}{2} \sum_{k=1}^{m} f''(\eta_k)(X_{t_k} - X_{t_{k-1}})^2$$

where $\eta_k(\omega)$ is an intermediate point.

Itô Formula. Idea of proof

$$\sum_{k=1}^{m} f'(X_{t_{k-1}})(X_{t_k} - X_{t_{k-1}}) = \sum_{k=1}^{m} f'(X_{t_{k-1}})(B_{t_k} - B_{t_{k-1}})$$

$$+ \sum_{k=1}^{m} f'(X_{t_{k-1}})(M_{t_k} - M_{t_{k-1}})$$

where:

$$B_t = \int_0^t K_s ds \text{ and } M_t = \int_0^t H_s dW_s$$

Itô Formula. Idea of proof

In regular calculus the term:

$$\frac{1}{2}\sum_{k=1}^{m} f''(\eta_k)(X_{t_k} - X_{t_{k-1}})^2$$

goes to zero as the norm of the partition goes to 0 since f'' is bounded (by Q) and then we can do:

$$\leq Q\|\Pi\|\frac{1}{2}\sum_{k=1}^{m}(x_k - x_{k-1}) \longrightarrow 0$$

Itô Formula. Idea of proof

But in this case that is not true. We can replace X by its two pieces again

$$\frac{1}{2}\sum_{k=1}^{m} f''(\eta_k)(X_{t_k} - X_{t_{k-1}})^2$$

$$= \frac{1}{2}\sum_{k=1}^{m} f''(\eta_k)(B_{t_k} + M_{t_k} - B_{t_{k-1}} - M_{t_{k-1}})^2$$

Developing the square we get 3 terms:

1. $(B_{t_k} - B_{t_{k-1}})^2$, which is a "calculus" term and so it goes to 0.
2. $(B_{t_k} - B_{t_{k-1}})(M_{t_k} - M_{t_{k-1}})$, which can also be bounded using the norm of the partition.
3. $(M_{t_k} - M_{t_{k-1}})^2$, which does not go to zero.

Itô Formula. Idea of proof

Proving that

$$\sum_{k=1}^{m} f''(\eta_k)(M_{t_k} - M_{t_{k-1}})^2 \longrightarrow \int_0^t f''(X_s)H_s^2 ds$$

is somewhat technical but intuitively clear since

$$(M_{t_k} - M_{t_{k-1}})^2 \sim H(t_k)^2(t_k - t_{k-1})$$