# Solving Black-Scholes' PDE

Black and Scholes equation:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0$$

To solve it we need final conditions. For a call:

$$C(S, T) = \max(S - K, 0)$$

First, let us introduce an integrating factor to get rid of the $rC$ term.

$$e^{-rt}\left(\frac{\partial C}{\partial t} - rC\right) = \frac{\partial(e^{-rt}C)}{\partial t}$$

Then, by calling $u = e^{-rt}C$ we get:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} + rS\frac{\partial u}{\partial S} = 0$$

Consider the "log-price" variable $x = \log(S)$ and let us reverse it into a forward equation by doing $s = T - t$.

$$\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial s}$$

$$\frac{\partial u}{\partial S} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial S} = \frac{\partial u}{\partial x}\frac{1}{S}$$

$$\frac{\partial^2 u}{\partial S^2} = \frac{\partial}{\partial S}(\frac{\partial u}{\partial x}\frac{1}{S}) = -\frac{1}{S^2}\frac{\partial u}{\partial x} + \frac{1}{S^2}\frac{\partial^2 u}{\partial x^2}$$

Replacing:

$$0 = -\frac{\partial u}{\partial s} + (r - \frac{1}{2}\sigma^2)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2}$$

We can now do another transform to kill the first order term in $x$.

Let us try the integrating factor trick again:

$$u(x,s) = e^{\alpha x + \beta s} v(x,s)$$

Then:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} e^{\alpha x + \beta s} + \alpha e^{\alpha x + \beta s} v(x,s)$$

$$\frac{\partial u}{\partial s} = \frac{\partial v}{\partial s} e^{\alpha x + \beta s} + \beta e^{\alpha x + \beta s} v(x,s)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} e^{\alpha x + \beta s} + \alpha^2 e^{\alpha x + \beta s} v(x,s) + 2\alpha \frac{\partial v}{\partial x} e^{\alpha x + \beta s}$$

Now we can choose $\alpha$ and $\beta$ so that we get rid of the first derivative.

Also, we can get rid of the $v(x,s)$ terms.

So, in the end we need to solve:

$$\frac{\partial v}{\partial s} = \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial x^2}$$

Which is the "traditional" heat equation.

Given an initial condition $v_0(x)$, the solution is obtained by convoluting $v_0$ with the gaussian kernel:

$$\Phi(x, s) = \frac{1}{\sqrt{4\pi sk}} e^{-\frac{x^2}{4ks}}$$

where $k$ in this case is $\frac{1}{2}\sigma^2$.

Let us see why:

To solve it we are going to use the Fourier Transform:

$$\hat{u}(\xi) = \int_{-\infty}^{\infty} u(x) e^{-ix\xi} dx$$

So, we can compute:

$$\widehat{\frac{\partial v}{\partial s}}(\xi, s) = \frac{1}{2}\sigma^2 \widehat{\frac{\partial^2 v}{\partial x^2}}(\xi, s)$$

So, we need to be able to compute the Fourier Transform of the partial derivatives of the function $v$.

1.
$$\widehat{\frac{\partial v}{\partial s}}(\xi, s) = \int_{-\infty}^{\infty} \frac{\partial v(x, s)}{\partial s} e^{-ix\xi} dx = \int_{-\infty}^{\infty} \frac{\partial (v(x, s) e^{-ix\xi})}{\partial s} dx$$

$$= \frac{\partial}{\partial s} \int_{-\infty}^{\infty} v(x, s) e^{-ix\xi} dx = \frac{\partial}{\partial s} \hat{v}(\xi, s)$$

2.
$$\widehat{\frac{\partial^2 v}{\partial x^2}}(\xi, s) = \int_{-\infty}^{\infty} \frac{\partial^2 v(x, s)}{\partial^2 x} e^{-ix\xi} dx$$

$$= -\int_{-\infty}^{\infty} \frac{\partial v(x, s)}{\partial x} (-i\xi) e^{-ix\xi} dx$$

$$= \int_{-\infty}^{\infty} v(x, s)(-i\xi)(-i\xi) e^{-ix\xi} dx$$

$$= (-i\xi)^2 \int_{-\infty}^{\infty} v(x, s) e^{-ix\xi} dx = (i\xi)^2 \hat{v}(\xi, s)$$

Therefore, going back to the equation:

$$\widehat{\frac{\partial v}{\partial s}}(\xi, s) = \frac{1}{2}\sigma^2 \widehat{\frac{\partial^2 v}{\partial x^2}}(\xi, s)$$

becomes:

$$\frac{\partial}{\partial s}\hat{v}(\xi, s) = \frac{1}{2}\sigma^2 (i\xi)^2 \hat{v}(\xi, s)$$

or:

$$\frac{\partial}{\partial s}\hat{v}(\xi, s) + \frac{1}{2}\sigma^2 \xi^2 \hat{v}(\xi, s) = 0$$

So, we got rid of the derivative with respect to $x$, we can use the "integrating factor trick" again:

$$(\frac{\partial}{\partial s}\hat{v}(\xi, s) + \frac{1}{2}\sigma^2 \xi^2 \hat{v}(\xi, s))e^{\frac{1}{2}\sigma^2 \xi^2 s} = 0$$

or:

$$\frac{\partial}{\partial s}(\hat{v}(\xi, s)e^{\frac{1}{2}\sigma^2\xi^2 s}) = 0$$

So, the function in the parenthesis depends only on $\xi$:

$$\hat{v}(\xi, s)e^{\frac{1}{2}\sigma^2\xi^2 s} = f(\xi)$$

or:

$$\hat{v}(\xi, s) = f(\xi)e^{-\frac{1}{2}\sigma^2\xi^2 s}$$

We now use the initial condition:

$$\hat{v}(\xi, 0) = \hat{v}_0(\xi) = f(\xi)$$

Then:

$$\hat{v}(\xi, s) = \hat{v}_0(\xi)e^{-\frac{1}{2}\sigma^2\xi^2 s}$$

To find the solution we now need to take the inverse of the Fourier Transform.

2 Facts about the Fourier Transform:

1. The Inverse Fourier Transform of a function $\hat{g}(\xi)$ is given by:

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(\xi) e^{ix\xi} d\xi$$

2. Consider two functions $f(x)$ and $g(x)$ we can define its "convolution":

$$(f * g)(x) = \int f(y) g(x - y) dy$$

Why would we want to do this?

For example, in probability, if we have two independent random variables $X$ and $Y$ with densities $f_X$ and $f_Y$, the density of the sum of the two is $f_X * f_Y$.

Taking the Fourier Transform of the convolution gives:

$$\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$$

In our case we have:

$$\hat{v}(\xi, s) = \hat{v_0}(\xi)e^{-\frac{1}{2}\sigma^2\xi^2 s}$$

So, applying the convolution result to:

$$\hat{f}(\xi) = \hat{v_0}(\xi)$$

and

$$\hat{g}(\xi) = e^{-\frac{1}{2}\sigma^2\xi^2 s}$$

we would find that:

$$v(x, s) = (f * g)(x, s)$$

We know that the inverse Fourier Transform of $\hat{v}_0(\xi)$ is $v_0(x)$.

But, what is the inverse Fourier Transform of $e^{-\frac{1}{2}\sigma^2\xi^2 s}$?
We have to compute it

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2\xi^2 s} e^{ix\xi} d\xi$$

Writing the exponent as:

$$-(\sqrt{\frac{1}{2}\sigma^2 s}\xi - i\frac{x}{\sqrt{2\sigma^2 s}})^2 - \frac{x^2}{2\sigma^2 s}$$

The first term is dealt with by comparing it to a normal distribution, the second term comes out from the integral.

The result is:

$$\frac{1}{\sqrt{2\pi\sigma^2 s}} e^{-\frac{x^2}{2\sigma^2 s}}$$