Itô's Product Rule

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What is wrong with this:

$$Y(t) = S(t)e^{-rt}$$

then

$$\frac{\partial Y}{\partial t} = S'(t)e^{-rt} + S(t)e^{-rt}(-r)$$

Itô's Product Rule

The derivative of $S(t)$ does not exist...

Moreover: if X, Y are two stochastic processes in general it is not true that $d(XY) = YdX + XdY$

Why? Think about this:

$$d(fg) = f(x+h)g(x+h) - f(x)g(x)$$

$$= f(x+h)g(x+h) + f(x)g(x+h) - f(x)g(x+h) - f(x)g(x)$$

$$= g(x+h)(f(x+h) - f(x)) + f(x)(g(x+h) - g(x))$$

$$= g(x+h)f'(x^*)h + f(x)g(x^{**})h = (g(x+h)f'(x^*) + f(x)g(x^{**}))h$$

So, the right hand term is just like a constant times h (the fact that g appears evaluated at $x+h$ is not a problem since g is smooth, so when h is small it is very close to $g(x)$).
Therefore we can divide it by h.

We could have also:

$$g(x+h)(f(x+h)-f(x))+f(x)(g(x+h)-g(x))$$

$$=(g(x+h)-g(x))(f(x+h)-f(x))+g(x)(f(x+h)-f(x))$$

$$+f(x)(g(x+h)-g(x))$$

and by the mean value theorem
$(g(x+h)-g(x))(f(x+h)-f(x))$ has order h^2.

Notice that we can do the argument without taking a derivative.

It is only at the end that, we see that we can divide by h and there are two terms that survive taking the limit as $h \to 0$.

We could say that a large part of calculus is the study of how functions change when the independent variables change.

In the case of the calculus of smooth functions we find that they move at the same pace as the independent variable(s).

In stochastic calculus they move faster.

Therefore, in stochastic calculus this is the point at which we stop, we do not take the limits (we already know that they do not exist)

Now, if:

$$dX = a_1 dt + b_1 dW, dY = a_2 dt + b_2 dW$$

What is $d(XY)$?

$$d(XY) = X(t+h)Y(t+h) - X(t)Y(t)$$

$$= X(t+h)Y(t+h) + X(t)Y(t+h) - X(t)Y(t+h) - X(t)Y(t)$$

$$= Y(t+h)(X(t+h) - X(t)) + X(t)(Y(t+h) - Y(t))$$

$$= (Y(t+h) - Y(t))(X(t+h) - X(t))$$

$$+Y(t)(X(t+h) - X(t)) + X(t)(Y(t+h) - Y(t))$$

The first term is just $dXdY$.

What is the order of $(Y(t+h) - Y(t))(X(t+h) - X(t))$?

This is just a simple generalization of the dW^2 that appears in all of the other cases that we studied.

So it is not h^2 as before because we have the $b_1 b_2 (dW)^2$ term...

Remark:

I did it in this way to show the differeces between regular calculus and stochastic calculus.

A more straightforward way to do it would be using Taylor, as usual. In this case, we would use the multidimensional version applied to the function $f(x, y) = xy$:

$$df(X, Y) = \frac{\partial f}{\partial x}(X, Y)dX + \frac{\partial f}{\partial y}(X, Y)dY$$

$$+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(dY)^2$$

$$+\frac{1}{2}\frac{\partial^2 f}{\partial x \partial y}dXdY + \frac{1}{2}\frac{\partial^2 f}{\partial y \partial x}dXdY$$

$$+ \text{ higher order terms}$$

Computing the derivatives we get:

$$df(X, Y) = d(XY) = YdX + XdY + dXdY$$