FM 5022
Hw12

1) When studying the Ho-Lee model we found that, in order to match bond prices, the function $\theta(u)$ must be $\frac{\partial f(0,u)}{\partial u} + \sigma^2 u$. Check that, with this choice of θ, the model coincides with the model obtained by using HJM with $\nu = 1$ and $\frac{\partial \sigma(t,T)}{\partial T} = \sigma$.

2) HJM is a model for instantaneous forwards, whereas LMM deals with forwards for a period $[t_k, t_{k+1}]$. Check that the SDE for the forwards in LMM coincides with HJM as $t_k - t_{k+1} \to 0$. In other words, check that

$$dF_k(t) = \zeta_k(t)F_k(t) \sum_{i=m(t)}^{k} \frac{\delta_i F_i(t)\zeta_i(t)}{1+\delta_i F_i(t)} dt + \zeta_k(t)F_k(t)dw$$

goes to:

$$df(t,T) = \sum_{i=1}^{\nu} \frac{\partial \sigma_i(t,T)}{\partial T} dW_i(t)$$
$$+ \left(\sum_{i=1}^{\nu} \frac{\partial \sigma_i(t,T)}{\partial T} \int_t^T \frac{\partial \sigma_i(t,s)}{\partial s} ds\right) dt$$

as $t_k - t_{k+1} \to 0$.

Remarks:

- Use $\nu = 1$.
- While writing this problem I noticed that the formula on the pdf I posted on Tuesday was missing the $F_k(t)$ inn the dt term.

3) In setting up a one-factor HJM, one thing we could decide to do is to have a volatility (the term that multiplies the brownian motion) which is (like in Black-Scholes) $\sigma f(t,T)$. In the notation of the handout $\frac{\partial \sigma(t,T)}{\partial T} = \sigma f(t,T)$.
By doing this we would be specifying (or trying to specify) a lognormal model for the instantaneous forwards. However:

a) What is the equation that the instantaneous forward rates follow in the risk neutral world in this case?
b) As you can see the drift is proportional to the square of the forward rate. This creates a problem: with this choice the forward rates explode. To see this solve the following ordinary differential equation

$$f'(t) = f^2(t)$$

and show that $f(t)$ blows up for some $t < \infty$.

This is one of the issues that forced researchers and practitioners to look for models that do not focus on instantaneous forwards.