Forward Rates

Zero-coupon bonds and zero-coupon rates

If $P(0, T_1)$ is the price of a zero-coupon bond maturing at time T_1 then the corresponding zero-coupon rate is defined by

$$P(0, T_1) = e^{-R(0,T_1)T_1}$$

Writing a zero rate as a function of the forward rates

The fwd rate between two future times T_1 and T_2 is the rate $f(0, T_1, T_2)$ that satisfies

$$e^{R(0,T_1)T_1} e^{f(0,T_1,T_2)(T_2-T_1)} = e^{R(0,T_2)T_2}$$

where $R(0, T_1)$ is the zero rate between now and time T_1.

Forward Rates

Therefore

$$f(0, T_1, T_2) = \frac{R(0, T_2)T_2 - R(0, T_1)T_1}{(T_2 - T_1)}$$

We know that a zero coupon bond maturing at time T_1 is worth

$$P(0, T_1) = e^{-R(0,T_1)T_1}$$

so we can rewrite the forward as

$$f(0, T_1, T_2) = -\frac{\ln P(0, T_2) - \ln P(0, T_1)}{(T_2 - T_1)}$$

Forward Rates

Letting T_2 approach T_1 we obtain

$$f(0,T) = -\frac{\partial \ln P(0,T)}{\partial T}$$

Then

$$P(0,T) = e^{-\int_0^T f(0,s)ds} = e^{-R(0,T_1)T_1}$$