From Random Variables to Random Processes

Random Processes

In probability theory we study spaces $(\Omega, \mathcal{F}, \mathcal{P})$ where Ω is the space, $\mathcal{F}$ are all the sets to which we can measure its probability and $\mathcal{P}$ is the probability.

Example: Toss a die twice.

$\Omega = \{(1,1),(1,2),...,(6,6)\}$

$\mathcal{F} =$ All the subsets of Ω.

$\mathcal{P}$ assigns the number $\frac{1}{36}$ to each pair.

We can do things like computing the probability of getting a 1 in the first trial or computing the probability of the sum of the two being larger than 3.

Each possible result of the experiment is denoted with the letter ω.

Random Processes

Now, we can also define a function

$$X : \Omega \longrightarrow \mathbf{R}$$

For example $X =$"first coordinate" or $X =$"sum of the two numbers".

X induces a probability on the real line by doing:

$$P(A) = \mathcal{P}(X \in A)$$

P is what is usually called the distribution of X.

From X we can measure expected value ($E(X)$), variance ($\mathrm{Var}(X)$), standard deviation, etc.

The numeric result of an experiment can be denoted by $X(\omega)$.

Random Processes

Now, suppose that you were to repeat the experiment every day.

Instead of X we would have X_t or $X(t)$.

After 1 year of doing so we would have created a whole function

If we were to do it again we would obtain:

$$X(t, \omega_2)$$

$X(t)$ is a stochastic process.

The result of an experiment $X(t, \omega)$ is a "random function".

Random Processes

From this X we can measure the expected value at a given time

$$E(X(t_0))$$

the expected value of the difference at two different times

$$E(X(t_1) - X(t_0))$$

the variance at a given time

$$Var(X(t_0))$$

etc.

In many examples Ω is not explicitly known

What is Ω in finance?

Brownian Motion

$B(t)$ is a Brownian Motion if:

1) $B(t)$ is cont in t.

2) $B(t+h) - B(t) \sim \mathrm{N}(0, \sqrt(h))$.

3) Indep. increments:

$$B(t_n) - B(t_{n-1}), ..., B(t_1) - B(t_0) \text{ are independent.}$$

Notice that, in particular,

$\mathrm{Var}\ (B(t_j) - B(t_{j-1})) = t_j - t_{j-1}$.

Now, consider an interval $I = [0, T]$ and consider a partition of I.

$$\Pi = t_0, t_1, ..., t_n \text{ with } 0 = t_0 < t_1 < ... < t_n = T$$

The mesh of the partition is defined as

$$\|\Pi\| = \max_j |t_j - t_{j-1}|$$

Given a function f we define the first variation and the quadratic variation as:

$$FV(f)(T) = \lim_{\|\Pi\| -> 0} \sum_{j=0}^{j=n} |f(t_j) - f(t_{j-1})|$$

$$< f > (T) = \lim_{\|\Pi\| -> 0} \sum_{j=0}^{j=n} |f(t_j) - f(t_{j-1})|^2$$

If f is differentiable:

$$f(t_j) - f(t_{j-1}) = f'(t_j^*)(t_j - t_{j-1})$$

and therefore

$$FV(f)(T) = \int_O^T |f'(t)| dt$$

and

$$< f > (T) = \lim_{\|\Pi\| -> 0} \sum_{j=0}^{j=n} |f'(t_j^*)|^2 |(t_j - t_{j-1})|^2$$

$$\leq \lim_{\|\Pi\| -> 0} \|\Pi\| \sum_{j=0}^{j=n} |f'(t_j^*)|^2 |(t_j - t_{j-1})| = 0$$

Fact:

$$< B > (t) = T$$

(This would say, in particular, that $B(t)$ is no differentiable)

Define $D_k = B(t_{k+1}) - B(t_k)$

The quadratic variation is the limit of expressions like:

$$Q_\Pi = \sum_{k=0}^{n-1} D_k^2$$

We want to prove that

$$\lim_{\|\Pi->0\|}(Q_\Pi - T) = 0$$

Consider

$$Q_\Pi - T = \sum_{k=0}^{n-1} D_k^2 - (t_{k+1} - t_k)$$

It is easy to see that the expected value is 0.

Because of independent increments

$$\mathrm{Var}(Q_\Pi - T) = \sum_{k=0}^{n-1} \mathrm{Var}(D_k^2 - (t_{k+1} - t_k)) = \sum_{k=0}^{n-1} \mathrm{E}(D_k^2 - (t_{k+1} - t_k))^2$$

$$= \sum_{k=0}^{n-1} \mathrm{E}(D_k^4 - 2D_k^2(t_{k+1} - t_k) + (t_{k+1} - t_k)^2)$$

$$= \sum_{k=0}^{n-1} (3(t_{k+1} - t_k)^2 - 2(t_{k+1} - t_k)^2 + (t_{k+1} - t_k)^2)$$

$$= \sum_{k=0}^{n-1} 2(t_{k+1} - t_k)^2 \leq 2\|\Pi\| T -> 0$$

Which is what we wanted

Now, notice that

$$E((B(t_{k+1}) - B(t_k))^2 - (t_{k+1} - t_k)) = 0$$

$$\text{Var}((B(t_{k+1}) - B(t_k))^2 - (t_{k+1} - t_k)) = 2(t_{k+1} - t_k)^2$$

(To see this consider $X \sim N(0, \sigma)$.

Then $E((X^2 - \sigma^2)^2) = E(X^4) - \sigma^4$, but $E(X^4) = 3\sigma^4$)

Or, which is the same:

$$E((B(t_{k+1}) - B(t_k))^2) = (t_{k+1} - t_k)$$

$$E(((B(t_{k+1}) - B(t_k))^2 - (t_{k+1} - t_k))^2) = 2(t_{k+1} - t_k)^2$$

(To see this consider $X \sim N(0, \sigma)$.

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Brownian Motion

Therefore, since $(t_{k+1} - t_k)^2$ is very small when $(t_{k+1} - t_k)$ is small :

$$(B(t_{k+1}) - B(t_k))^2 \sim (t_{k+1} - t_k)$$

In differential notation:

$$dB(t)dB(t) = dt$$

Remind: Riemann-Stieltjes

In calculus we study Riemann integrals:

$$\int_a^b f(x)dx = lim \sum f(x_i)(x_i - x_{i-1})$$

where the x_is form a partition of the interval $[a, b]$ and the limit is taken as the norm of the partition goes to zero.
In that sum $(x_i - x_{i-1})$ represent the weight (or measure) we give to the interval $[x_i, x_{i-1}]$.
There is a generalization of that concept called Riemann-Stieltjes integral in which the weight assigned to each interval is given by a transformation $g(x)$.

$$\int_a^b f(x)dg(x) = lim \sum f(x_i)(g(x_i) - g(x_{i-1}))$$

Remind: Riemann-Stieltjes

For us to be able to do that $g(x)$ has to satisfy some conditions.

For example if g is an increasing function we can do it.

In that case g represents a "deformation" of the original homogeneous measure.

A general class of functions for which this can be done is formed by the functions that have finite first variation.

Back to Brownian Motion

Given a function f how do we compute $df(B(t))$?
In calculus we do

$$\frac{d}{dt}f(B(t)) = f'(B(t))B'(t)dt$$

In differential notation

$$df(B(t)) = f'(B(t))B'(t)dt = f'(B(t))dB(t)$$

But now $B(t)$ is not differentiable, in particular $B(t+h) - B(t)$ is "too big". However, we know that $(B(t+h) - B(t))^2 \sim h$ so we try adding an extra term to the Taylor expansion:

So, let us think about the chain rule in both cases:

Let us take a smooth function f which we are going to compose with:

1. a function g also smooth.
2. a brownain motion B.

$$f(g(t_k)) - f(g(t_{k-1})) = f'(\xi)(g(t_k) - g(t_{k-1}))$$

where ξ is between $g(t_k)$ and $g(t_{k-1})$. Let us now divide both sides by $(t_k - t_{k-1})$ and let t_k and t_{k-1} be very close to each other to obtain:

$$(f \circ g)'(t_k) = f'(g(t_k))g'(t_k)$$

If, instead of developing up to order 1 we had developed up to order 2:

$$f(g(t_k)) - f(g(t_{k-1})) = f'(g(t_{k-1}))(g(t_k) - g(t_{k-1}))$$

$$+\frac{1}{2}f''(\xi)(g(t_k) - g(t_{k-1}))^2$$

Again, dividing both sides by $(t_k - t_{k-1})$ and letting t_k and t_{k-1} we see that the second term vanishes (why?). So, the result is the same.

Now, what happens if instead of g we have B?

Suppose that we stop at order 1:

$$f(B(t_k)) - f(B(t_{k-1})) = f'(\xi)(B(t_k) - B(t_{k-1}))$$

We can't divide by $(t_k - t_{k-1})$ as before ($\frac{B(t_k)-B(t_{k-1})}{(t_k-t_{k-1})}$ blows up in the limit).

If we try going up to order 2:

$$f(B(t_k)) - f(B(t_{k-1})) = f'(B(t_{k-1}))(B(t_k) - B(t_{k-1}))$$

$$+\frac{1}{2}f''(\xi)(B(t_k) - B(t_{k-1}))^2$$

We still can't divide. But, what we can do is to make t_k and t_{k+1} be very close, sum over al the $t_k's$ and see if the two terms on the right make sense. It turns out that this can be done (we will do this soon).

So, in the end:

$$df(B(t)) = f'(B(t))dB(t) + \frac{1}{2}f''(B(t))(dB(t))^2$$

$$= f'(B(t))dB(t) + \frac{1}{2}f''(B(t))dt$$

In integral form

$$f(B(T)) - f(B(0)) = \int_0^T f'(B(t))dB(t)$$

$$+\frac{1}{2}\int_0^T f''(B(t))dt$$

Brownian Motion

Example: $f(x) = \frac{1}{2}x^2$

$$f'(x) = x, f''(x) = 1$$

$$\frac{B^2(T)}{2} = \int_0^T B(t)dB(t) + \frac{1}{2}T$$

We should compare this to

$$\int_0^T xdx = \frac{T^2}{2}$$

...so, in stochastic calculus, we have an extra term

Remark:

When one defines the stochastic integral one finds that

$$E(\int_0^T B(s)dB(s)) = 0$$

One the other hand we know that $E(\frac{B^2(T)}{2}) = \frac{T}{2}$.

So...if we did not have the extra term we would be in trouble.

Itô

Let us assume that the return of stocks is governed by:

$$\frac{S_{t+1} - S_t}{S_t} = \mu t + \phi \text{ where } \phi \sim N(0, \sigma)$$

In continuous time:

$$\frac{dS}{S} = \mu dt + \sigma dB.$$

How do I solve that? (how do I find S_t?)

If we were talking regular calculus the solution would be log(S).
So, let's try the same solution:
Using Taylor:

$$log(S_t) = log(S_0) + \frac{1}{S_0} dS_0 - \frac{1}{2}\frac{1}{S_0^2}(dS_0)^2$$

I can replace $dS_0 = S_0 \mu dt + S_0 \sigma dB$.

Also, $(dS_0)^2 = S_0^2 \mu^2 dt^2 + S_0^2 \sigma^2 dB^2 + 2\mu\sigma S_0^2 dtdB$

However, I know that $dB^2 \sim dt$.

So, the term containing dB^2 is the biggest of them three.

If I now discard all the terms smaller than dt we end up with:

$$log(S_t) = log(S_0) + \frac{1}{S_0}(S_0 \mu dt + S_0 \sigma dB) - \frac{1}{2} S_0^2 \sigma^2 dt$$

or

$$log(S_t) = log(S_0) + (\mu dt + \sigma dB) - \frac{1}{2}\sigma^2 dt$$

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B(t)}$$

Itô

In general, from time t to time $t+h$ the solution evolves as:

$$S_{t+h} = S_t e^{(\mu - \frac{1}{2}\sigma^2)h + \sigma(B(t+h) - B(t))}$$

But we know that $B(t+h) - B(t) \sim N(0, \sqrt{h})$. The we can rewrite as:

$$S_{t+h} = S_t e^{(\mu - \frac{1}{2}\sigma^2)h + \sigma\sqrt{h}X} \text{ where } X \sim N(0, 1)$$