Duration

Duration

It is a measure to compare bonds (among other things).

It provides an estimate of the volatility or the sensitivity of the market value of a bond to changes in interest rates.

There are two very closely related measures: Macaulay Duration and Modified Duration.

Macaulay Duration

The idea is to cook up a number that tells us information about the time in which cash flows are received.

We could use the maturity of the bond.

This is not satisfactory since we are ignoring all the possible coupons paid previous to the final cash flow.

Macaulay proposed a measure that weights the times in which the cash flows are received.

Macaulay Duration

Recall the pricing formula for a bond:

$$B = \sum_{t=1}^{m} \frac{C_t}{(1+r)^t}$$

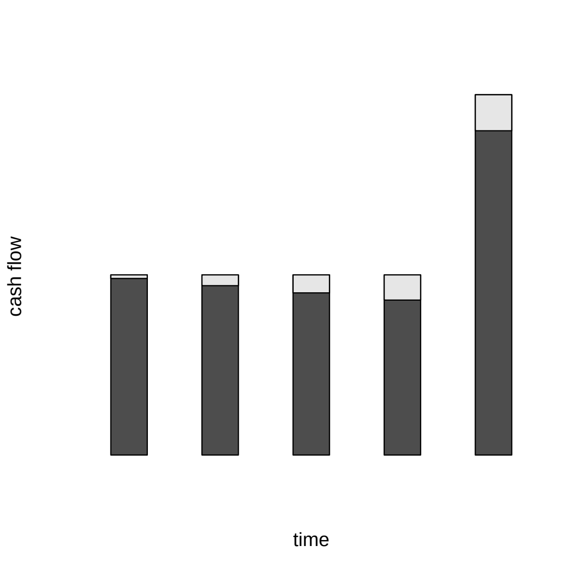
where C_t is the cash flow paid at time t.
Using continuous compounding it reads:

$$B = \sum_{t=1}^{m} C_t e^{-rt}$$

Then, Macaulay's formula is:

$$D = \frac{\sum_{t=1}^{m} \frac{tC_t}{(1+r)^t}}{\sum_{t=1}^{m} \frac{C_t}{(1+r)^t}}$$

Weighting the Cash Flows



Macaulay Duration

Notice that the denominator is just the price of the bond, so we can rewrite

$$D = \sum_{t=1}^{m} t\left(\frac{C_t}{B(1+r)^t}\right)$$

With continuous compounding:

$$D = \frac{\sum_{t=1}^{m} tC_t e^{-rt}}{B}$$

Macaulay Duration

For example, what is the duration of a zero-coupon bond?

$$D = T(\frac{C_T}{B(1+r)^T}) = T$$

so, it is the maturity.

Macaulay Duration

Sensitivities

How does the duration change when we extend maturity? It increases

How does the duration change when interest rates change? It decreases

How does the duration change when time goes by? It decreases

How does the duration change when a coupon is paid? It jumps up a bit.

Macaulay Duration

Let us now take the definition using continuous compounding and take the derivative of B with respect to the interest rate

$$\frac{\partial B}{\partial r} = -\sum_{t=1}^{m} C_t t e^{-rt} = -BD$$

So, changes in prices due to small parallel shifts to the yield curve are very closely related to the Duration.

We can rewrite

$$\frac{\Delta B}{\Delta r} = -BD$$

Example

Consider a three-year 10% coupon bond with face value of \$100. Suppose that the yield is 12% per anuum with continuous compounding. What is the value of the bond? If the yield moves by .1% what is the change in the value of the bond? Solution: Let

us compute the value of the bond and its duration

Time	*Payment*	*PresentValue*	*Weight*	*Time* X *Weight*
.5	5	4.709	0.050	0.025
1.0	5	4.435	0.047	0.047
1.5	5	4.176	0.044	0.066
2.0	5	3.933	0.042	0.083
2.5	5	3.704	0.039	0.098
3.0	105	73.256	0.778	2.333
Total	130	94.213	1.000	2.653

Example

Therefore the price of the bond is \$94.213 and the duration is 2.653.

Then

$$\Delta B = -94.213 \text{ X } 2.654 \Delta r = -250.04 \Delta r$$

So, if r goes from .12 to .121, $\Delta r = .001$ and $\Delta B = -0.25$ which changes the value of the from 94.213 to $94.213 - 0.25 = 93.963$.

We could recompute tha table with 12% replaced by 12.1% and verify that this is the price that we obtain.

What have we done?

Example

Two things.
1)Recall from calculus Taylor's expansion

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2} + ...$$

So, if we know that value of f at x and we also now the derivatives we can approximate $f(x+h)$. If h is small we can approximate by using only the first term

$$f(x+h) = f(x) + f'(x)h$$

(We are going to use this a lot during the class)
2) We have identified the derivative of the price of a bond with respect to the yield.
Then, putting 1) and 2) together we can approximate the new price of the bond when the yield changes by a little bit.

Application to "curve trading"

Following with the example, suppose that we observe a bond maturing in 3.5 years, also paying 10% coupon, but yielding 15%. We think that 3% difference in the yield of both bonds is too much. So, we decide to "play the spread". We would have to buy the 3.5-year bond and sell the 3-year bond. The question is: **in what ratio**?

Playing the spread means that we want to be indifferent to parallel moves in the yield curve, in other words, if tomorrow the 3-year bond goes down in value so that it yields 14% and the 3.5-year bond also goes down in value so that it yields 17%, we would want to be flat on the trade.

As we know the duration gives us a measure of how much the percentage change in the value of a bond changes when yields change.

Application to "curve trading"

We would like (assume I have included the notional size in the B's)

$$\Delta B_3 = \Delta B_{3.5}$$

Then, we need

$$B_3 D_3 \Delta r_3 = B_{3.5} D_{3.5} \Delta r_{3.5}$$

as we want to equate the differences when $\Delta r_3 = \Delta_{3.5}$ we obtain

$$B_3 D_3 = B_{3.5} D_{3.5}$$

Convexity

Duration gives us a first-order approximation to the change in bond prices as yields change.
But, the bond price as a function of the yield is not linear.
So, when yields changes are not very small the duration does not give enough information.

$$\Delta B = \frac{\partial B}{\partial y}\Delta y + \frac{1}{2}\frac{d^2 B}{dy^2}\Delta y^2$$

Dividing by B

$$\frac{\Delta B}{B} = -D\Delta y + \frac{1}{2}C\Delta y^2$$

If we define the convexity as $\frac{1}{B}\frac{d^2 B}{dy^2}$