Derivation of the Heat Equation

Consider a particle x, with position at time t denoted by $x(t)$.

x moves like a random walk, for example:

$$x(t+\Delta t) = x(t) + \ell(t)$$

The size of the step ℓ follows a density $\chi(\ell)$ which has mean 0 and standard deviation a. Then;

$$\int \chi(z) dz = 1$$

$$\int z\chi(z) dz = 0$$

$$\int z^2\chi(z) dz = a^2\Delta t$$

What is the probability distribution of the location of x at time $t + \Delta t$ given that we know the density at time t?

Let us call $\rho(x', t)$ and $\rho(x, t + \Delta t)$ both those densities.

Now, If the particle is at x' at time t, the step that has to take to be at x at time $t + \Delta t$ is $x - x'$.

Considering all the possible locations at time x':

$$\rho(x, t + \Delta t) = \int_{-\infty}^{\infty} \rho(x', t)\chi(x - x')dx' = \int_{-\infty}^{\infty} \rho(x - z, t)\chi(z)dz$$

If we now develop ρ around x and use Taylor, we can approximate:

$$\rho(x-z,t) = \rho(x,t) - z\frac{\partial \rho}{\partial x} + \frac{1}{2}z^2\frac{\partial^2 \rho}{\partial x^2}$$

Which gives:

$$\begin{aligned}
\rho(x,t+\Delta t) &\approx \int_{-\infty}^{\infty}(\rho(x,t) - z\frac{\partial \rho}{\partial x} + \frac{1}{2}z^2\frac{\partial^2 \rho}{\partial x^2})\chi(z)dz \\
&= \rho(x,t)\int \chi(z)dz - \frac{\partial \rho}{\partial x}\int z\chi(z)dz \\
&\quad + \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2}\int z^2\chi(z)dz \\
&= \rho(x,t) + \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2}a^2\Delta t
\end{aligned}$$

Rewriting:

$$\rho(x, t + \Delta t) - \rho(x, t) = \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2} a^2 \Delta t$$

Dividing both sides by Δt the left hand side is approximately the derivative with respect to t:

$$\frac{\partial \rho}{\partial t} = c \frac{\partial^2 \rho}{\partial x^2}$$

where $c = \frac{1}{2}a^2$.

What if we had developed to a higher order?

$$\rho(x-z,t)=\rho(x,t)-z\frac{\partial\rho}{\partial x}+\frac{1}{2}z^2\frac{\partial^2\rho}{\partial x^2}-\frac{1}{3!}z^3\frac{\partial^3\rho}{\partial x^3}+\frac{1}{4!}z^4\frac{\partial^4\rho}{\partial x^4}$$

The third term would vanish just like the first one did.

The fourth would end up giving a $3\,a^4\,(\Delta t)^2$, which would vanish when we divide by Δt both sides.