Black-Scholes

The starting point is the stochastic differential equation

$$\frac{dS}{S} = \mu dt + \sigma dW.$$

which we solve the equation by applying Taylor to $log(S)$ (Itô's lemma) to find:

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma\epsilon\sqrt{t}}$$

where ϵ is a normal random variable with mean 0 and variance 1. With this model we try to describe the behavior of assets like the one we saw on the handout from the first class (CCMP, SPX, CL1, GC1, etc)

Valuing Options

(The idea of what we are going to do is very similar to what we did when studying binomial trees, we will set up a riskless portfolio containing the option and the share)
Suppose that C represents the value of a call on S. C is a function of S, therefore we can try to find the equation for C just as we did for the log(S) (Itô's lemma)

$$dC = \frac{\partial C}{\partial S} dS + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} dS^2 + \frac{\partial C}{\partial t} dt$$

As before

$$\frac{dS}{S} = \mu dt + \sigma dW \text{ and } (dS)^2 \sim S^2 \sigma^2 dt$$

then

$$dC = \sigma S \frac{\partial C}{\partial S} dW + (\mu S \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t}) dt$$

Valuing Options

Idea: dW is the only stochastic variable there. dW represents the risk, the unknown. But the same dW appears in the equation for S and in the equation for C. So, we could try to cancel the risk by trading both C and S (not one for one though).
Form a Portfolio: $\Pi = C - \Delta S$

$$d\Pi = (\mu S \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t} - \mu \Delta S)dt$$
$$+\sigma S(\frac{\partial C}{\partial S} - \Delta)dW.$$

If $\Delta = \frac{\partial C}{\partial S}$ then dW dissapears!!

$$d\Pi = (\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t})dt$$

Π is riskless, therefore it must return the riskless rate r (otherwise there is an arbitrage).

Valuing Options

So, $d\Pi = r\Pi dt$ then

$$r\Pi = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t}$$

and

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0$$

This is the Black and Scholes equation

Notice:

1) This equation is not stochastic, if we know how to solve partial differential equations we know how to solve it.

2) μ does not appear.

3) The argument works also for a put, actually it works for any function that depends on S.

We have said that the diff equation is valid for any function of S (not just for a call or a put)

Remember that to solve a differential equation we need also initial conditions, for example

$$\frac{dy}{dx} = 1$$

has infinitely many solutions since any $y(x) = x + a$ satisfies it. But if we add the restriction that $y(0) = 0$ we obtain $a = 0$ and therefore $y(x) = x$

Valuing Options

In our case we have

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0$$

What is the initial condition? (in this case it will be called final condition)

In the case of a call it is $C(S, T) = max(0, S_T - K)$, in the case of a put it will be $P(S, T) = max(0, K - S_T)$. In the case of a forward contract written on S it will be $f(S, T) = S_T - K$.

Valuing Options

In the case of a froward contract, we know that the value at time t is

$$f = (F - K)e^{-r(T-t)} \text{ where } F = Se^{r(T-t)}$$

and K is the delivery price, then

$$f = S - Ke^{-r(T-t)}$$

Does f satisfy the equation? It should. Let's check it:

$$\frac{\partial f}{\partial t} = -rKe^{-r(T-t)}, \frac{\partial f}{\partial S} = 1, \frac{\partial^2 f}{\partial S^2} = 0$$

The equation becomes:

$$-rKe^{-r(T-t)} + rS = rf$$

which is right.

Valuing Options

Going back to calls and puts: the equation can be solved with the final condition for both cases and the solutions are:

$$c = S_0 N(d_1) - X e^{-rT} N(d_2)$$

and

$$p = X e^{-rT} N(-d_2) - S_0 N(-d_1)$$

where $N(x)$ represents the cumulative standard normal at x (so it is the probability that the standard normal is less than x).

$$d_1 = \frac{ln(S_0/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{ln(S_0/X) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$$

Valuing Options

See what happens if $S_0 > X$ and $T- > 0$, what is the value of a call? what about a put? what if $S_0 < X$?

Remark: we did all this for a stock that does not pay dividends

Example: Suppose that NOK does not pay dividends and it is trading at \$25. I would like to buy a 6-month put with strike $X = 23$. The interest rate is 0% and the volatility is $\sigma = 40\%$. How much will I pay?

If the vol is 80% instead will I pay more or less? What if I prefer a

1 year option? Should I prefer to double the vol or to double the

time?

Risk Neutral Valuation

There is another way to arrive at Black-Scholes. This way is purely probabilistic and stems from the fact that the expected return does not appear into the equation.

Recall that

$$\frac{dS}{S} = \mu dt + \sigma dW.$$

Let's assume that instead of that equation S satisfies

$$\frac{dS}{S} = rdt + \sigma dW.$$

in other words, the expected return is the risk free rate.

Risk Neutral Valuation

Then, we can arrive at the price of the option by computing the expected value of the payoff (and discounting to the present) So,

for example the value of a call will be

$$c = e^{-rT}\hat{E}(max(S_T - K, 0))$$

the hat on top of the expectation denotes the fact that we are not using the true process S but rather modifying its expected return. (proving this involves another theorem from stochastic calculus called Girsanov's Theorem)

This is fairly straightforward since

$$\hat{E}(max(S_T - K, 0)) = \int_K^{\infty}(s - K)f(s)ds$$

Risk Neutral Valuation

How do we find the value of an option by simulating?

Remember that the solution to

$$\frac{dS}{S} = rdt + \sigma dW.$$

is

$$S_T = S_0 e^{(r-\sigma^2/2)T+\sigma\epsilon\sqrt{T}}$$

Therefore we can take a bunch of numbers from a standard normal random number generator and in turn create a bunch of possible values for S_T.
Let us denote those values by $x_i, i = 1, .., n$

We can then estimate

$$\hat{E}(max(S_T - K, 0)) = \frac{1}{n}\sum_{i=1}^{n} max(x_i - K, 0)$$

What vol do we enter into the equation?

The volatility is a parameter of the model. We know that we can estimate since it is the stddev of the returns. But, depending on whether we use 20 days, 40 days, etc we will get different estimates, which is the right one?

Actually....none of them.

The way we use the model is in reverse.
The observed price is the option price. So, given the option price we back out the volatility that the market is assigning to the stock. This is called **implied volatility**.

Question: Suppose that you observe two options trading on the same stock, one of them with $T = 1$ year and the other with $T = 2$ years. You look at the prices and compute the implied vols, shouldn't they be the same?

Adding Dividends

If S pays a continuous dividend yield q then the change in wealth of a long position in S is

$$dS + qSdt$$

Therefore the portfolio: $\Pi = C - \Delta S$
changes as

$$d\Pi = (\mu S \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t} - \mu \Delta S)dt$$

$$+\sigma S(\frac{\partial C}{\partial S} - \Delta)dW - \Delta qSdt$$

Adding Dividends

Again we set $\Delta = \frac{\partial C}{\partial S}$:

$$d\Pi = (\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t})dt - \frac{\partial C}{\partial S} qSdt$$

Π is riskless, therefore it must return the riskless rate r (otherwise there is an arbitrage).
So, $d\Pi = r\Pi dt$ then

$$r(C - \frac{\partial C}{\partial S} S) = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t} - \frac{\partial C}{\partial S} qS$$

and

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - q)S\frac{\partial C}{\partial S} - rC = 0$$

This is the Black and Scholes equation for a stock that pays a continuous div yield q