Binomial Trees: Additional Notes

We have seen that, using the hedging (delta-hedging, no-arbitrage) argument we obtain:

$$f = S_0\Delta - (S_0 u\Delta - f_u)e^{-rT}$$

where

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$$

So, we can now try to rearrange the terms to see what is f today as a function of its own price one period later (f_u, f_d).

If we substitute we get:

$$f = \frac{f_u - f_d}{u - d} - (\frac{u(f_u - f_d)}{u - d} - f_u)e^{-rT}$$

$$f = e^{-rT}(e^{rT}\frac{f_u - f_d}{u - d} - \frac{uf_u - uf_d - uf_u + df_u}{u - d})$$

$$f = e^{-rT}(f_u\frac{e^{rT} - d}{u - d} + f_d\frac{u - e^{rT}}{u - d})$$

Calling $p = \frac{e^{rT} - d}{u - d}$ we get

$$f = e^{-rT}(f_u p + f_d(1 - p))$$

Matching volatility

Choosing u and d have to do with the distribution of S.

Suppose that the time step is Δt.

The distribution of $S_{\Delta t}$ in a binomial tree is...binomial.
It takes the values:

- $S_0 u$ with probability p^*.
- $S_0 d$ with probability $1 - p^*$.

Matching volatility

The continuous compounded return is:

$$S_{\Delta t} = S_0 e^{\mu \Delta t}$$

In the binomial tree

$$p^* S_0 u + (1 - p^*) S_0 d = S_0 e^{\mu \Delta t}$$

From the we can get $p^* = \frac{e^{\mu \Delta t} - d}{u - d}$

Matching volatility

Remember: the volatility is defined as the standard deviation of the (cont) returns.

Question: If σ_1 is the volatility in year 1 and σ_2 is the volatility in year 2. What is the volatility for the whole period? (years 1 and 2 together).

$$ln(S_2/S_0) = ln(S_2/S_1 * S_1/S_0) = ln(S_2/S_1) + ln(S_1/S_0)$$

The returns corresponding to years 1 and 2 are independent.

If X, Y are independent random variables then

$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$.

Matching volatility

Then:

$$\sigma_{1+2}^2 = \sigma_1^2 + \sigma_2^2$$

Now, if $\sigma_1 = \sigma_2$ then $\sigma_{1+2} = \sqrt{2}\sigma$.

So, assuming independence of returns in non-overlapping periods we get that the volatility scales with the square root of time.

Note: The volatility is always quoted in annualized terms, so when using it for a period different from 1 year we need to account for the time difference. (This is just like with interest rates)

Matching volatility

The, going back to our binomial tree, if the volatility is assumed to be σ then the volatility in 1 period (length is Δt) will be

$$\text{stdev } \ln(S_{\Delta t}/S_0) = \sigma\sqrt{\Delta t}$$

What is the variance in terms of the tree?

$$p^* u^2 + (1 - p^*)d^2 - (p^* u + (1 - p^*)d)^2 = \sigma^2 \Delta t$$

Matching volatility

Together with

$$p^* S_0 u + (1 - p^*) S_0 d = S_0 e^{\mu \Delta t}$$

(which we found before) we have two equations to find u and d. However, as we see, p^* representing the real world probabilities enters into the equations. As we know, p^* is irrelevant to price options. All this analysis can be done using p instead of p^* (or, which is saying the same thing: in the risk neutral world instead of the real world).

In which case, from that first equation, we obtain:

$$p = \frac{e^{r \Delta t} - d}{u - d}$$

So, all in all, we have 2 equations to find 3 parameters p, u, d.

Remark

Notice that the fact that we know that

$$p = \frac{e^{r\Delta t} - d}{u - d}$$

says that for any S_0 we can multiply through to get:

$$pS_0 u + (1-p)S_0 d = S_0 e^{r\Delta t}$$

In other words, the expected return in the risk-neutral world is the risk-free rate.

Matching volatility

We nee dot fix one more condition. There are two popular choices for this:

- $p = 1/2$
- $u = 1/d$

Matching volatility: Case $u = 1/d$

Remember the two equations:

$$pu + (1-p)d = e^{r\Delta t}$$

$$pu^2 + (1-p)d^2 - (pu + (1-p)d)^2 = \sigma^2 \Delta t$$

Using the formula for p given by the first equation and replacing into the second one we get:

$$e^{r\Delta t}(u+d) - ud - e^{2r\Delta t} = \sigma^2 \Delta t$$

Now, using Taylor's approximation:$e^x = 1 + x + x^2/2 + ...$ and ignoring the terms containing powers of Δt higher than 1 we can see that $u = e^{\sigma\sqrt{\Delta t}}(\cong 1 + \sigma\sqrt{\Delta t} + 1/2\sigma^2\Delta t)$ is a solution. Then $d = e^{-\sigma\sqrt{\Delta t}}$.

Matching volatility: Case $p = 1/2$

In this case u and d are different:

$$u = e^{(r-\sigma^2/2)\Delta t+\sigma\sqrt{\Delta t}}$$

$$d = e^{(r-\sigma^2/2)\Delta t-\sigma\sqrt{\Delta t}}$$