Binomial Model, General Multiperiod Setting

Ref: Baxter-Rennie. Chapter 2

Assume that there is two traded assets in the world a **stock** and a **bond**.
(The bond represents the time-value of money)

Suppose that only two things can happen between time $t_0 = 0$ and $t_1 = \delta t$.

The stock can move up with probability p and down with probability $1 - p$.

Let us call s_1 the value of the stock right now, s_3 the "up" value and s_2 the "down" value.

Binomial Model

The bond moves in a deterministic way. If the interest rate is represented by r then \$1 at time 0 grows to to $e^{r\delta t}$ at time t_1.

Goal: Finding fair value for general payoffs at time t_1. For example nonlinear functions of the stock price.

For example a future is a (linear) function of the underlying.

How did we assigned a value to a futures contract?

We "reconstructed" it by selling a bond and using the cash gotten to buy the stock.

Replicating

Strategy: Find a replicating portfolio.

If two portfolios have the same value at some point in the future the have to be worth the same today.

We know that it is easy to replicate a future. Can we replicate any $f(S)$?

Denote our portfolio by (ϕ, ψ) where ϕ the amount of stock that we hold and ψ is the amount of bonds that we hold.

The porfolio today has value $\phi s_1 + \psi$.

and at time δt will have one the following values (depending on the value of the stock):

$$\phi s_3 + \psi e^{r\delta t}$$

$$\phi s_2 + \psi e^{r\delta t}$$

Replicating

If the value of the derivative at $t = \delta t$ is $f(3) = f(s_3)$ and $f(2) = f(s_2)$ we can solve for ϕ and ψ:

$$\phi = \frac{f(3) - f(2)}{s_3 - s_2}$$

$$\psi = e^{-r\delta t}(f(3) - \frac{(f(3) - f(2))s_3}{s_3 - s_2})$$

Therefore, we have synthesized the derivative.

Note: We had no restrictions on f.

Can we use this fact to find the current price of f ?

Arbitrage

Claim: the value of f today is

$$f(1) = s_1(\frac{f(3) - f(2)}{s_3 - s_2}) + e^{-r\delta t}(f(3) - \frac{(f(3) - f(2))s_3}{s_3 - s_2})$$

How do we prove this?

By arbitrage (a very powerful proving tool). Doing anything else will make us lose money.

If we had tried to price using the expected value of the payoff we would have gotten the wrong price.

Arbitrage

However:

If we rewrite the formula we got for $f(1)$:

$$f(1) = e^{-r\delta t}(f(3)\frac{(e^{r\delta t}s_1 - s_2)}{s_3 - s_2} + f(2)\frac{(s_3 - e^{r\delta t}s_1)}{s_3 - s_2})$$
$$= e^{-r\delta t}((1-q)f(2) + qf(3))$$

where

$$q = \frac{(e^{r\delta t}s_1 - s_2)}{s_3 - s_2}$$
$$1 - q = \frac{(s_3 - e^{r\delta t}s_1)}{s_3 - s_2}$$

Arbitrage

Obvious example: what if $f(S) = S$?

$$s_1 = e^{-r\delta t} E_Q(S_{\delta t})$$

where Q means that we are taking the expected value with respect of the probabilities given by q instead of p (or any other).

This type of formulae appear a lot in the derivatives literature.

Some Remarks and Concepts:

1) Notice that on a tree the probabilities vary with the node.

2) The set of (true) probabilities denoted by p were not necessary to price the claim f.

Now the concepts:

3) A **filtration** $(\mathcal{F}_i)$ is the history up until time i. In other words, it is the mathematical representation of the information known at time i.

4) A *claim* X_T on the tree is a function of the nodes at time T, or, which is the same, a function of $\mathcal{F}_T$.

Binomial Representation Theorem

5) The conditional expectation operator $E_Q(\cdot|\mathcal{F}_i)$ extends the usual notion of expected value.

$E_Q(\cdot|\mathcal{F}_i) =$ expected value of X given the information up until time i.

6) A *previsible* process $\phi = \phi_i$ is a process on the same tree whose value at any given node at time i depends only on the history up to time $i-1$ (i.e. depends on $\mathcal{F}_{i-1}$).

7) A process S is a *martingale* with respect to a probability P and a filtration $\mathcal{F}_i$ if

$$E_P(S_j|\mathcal{F}_i) = S_i, \text{ for all } i \leq j$$

Notice the dependence on the probabiliy P.

Filtrations

Suppose we have a tree in which S starts at \$100 and it moves up and down twice with u and d.

What are $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_2$?

What is $E_Q(S_2|\mathcal{F}_0)$?

What is $E_Q(S_2|\mathcal{F}_1)$?

Martingales

In simple words a martingale is a process that has no drift.
Example: Toss a fair coin (probability of heads and tails is both 1/2).
Define the random variable X so that:

$$X(\text{heads}) = 1, \; X(\text{tails}) = -1.$$

The expected value of X under the probabiliy that assigns probabilities 1/2 to each possible outcome is 0.

$$E(X) = X(heads) * p(heads) + X(tails) * p(tails)$$
$$= 1 * \frac{1}{2} + (-1) * \frac{1}{2}$$

If X_1 and X_2 represent the results of tossing the coin twice, what is the expected value of $X_1 + X_2$?

$$E(X_1 + X_2) =$$

$$X_1(heads) * p(heads) * X_2(heads) * p(heads) +$$

$$X_1(heads) * p(heads) * X_2(tails) * p(tails) +$$

$$X_1(tails) * p(tails) * X_2(heads) * p(heads) +$$

$$X_1(tails) * p(tails) * X_2(tails) * p(tails) = 0$$

Now, let's repeat this experiment and construct the process

$$S_n = \sum_{i=1}^{n} X_i$$

What is $E_P(S_j|\mathcal{F}_i)$? It is 0, therefore it is a martingale.
What if

$$S_n = \sum_{i=1}^{n} (1 + X_i)?$$

It is not a martingale.

Binomial Representation Theorem

Suppose that a binomial price process S is a Q martingale. If N is any other Q-martingale then there exists a previsibel process ϕ such that

$$N_i = N_0 = \sum_{k=1}^{i} \phi_k \Delta S_k$$

where $\Delta S_i = S_i - S_{i-1}$ is the change in S from time $i-1$ to time i.

So, what we are saying is that we can transform S into N with ϕ which is known in advance.

Financial application: Pricing by arbitrage

We will now use the theorem to price derivatives.
First, we have been talking about martingales but neither the stock S nor the claim X are martingales. S is a process but no necessarily a martingale and X is just a random variable, not even a process.
Fact: Given a random variable X the process $E_i = E_Q(X|\mathcal{F}_i)$ is a martingale. This is true due to

$$E_Q(E_Q(X|\mathcal{F}_j)|\mathcal{F}_i) = E_Q(X|\mathcal{F}_i), \; i \leq j.$$

This can be done with any probability so, if we find a probability Q under which S is a martingale we can make X into a martingale using the same probability Q.

Pricing by arbitrage

We also have a cash bond B_i representing the time value of money.
The bond B_i is a previsible process (it is actually deterministic).
We assume that $B_0 = 1$.
We will call:
1) Discount process: B_i^{-1}.
2) Discounted price process $Z_i = B_i^{-1} S_i$.
3) Discounted claim: $B_T^{-1} X$.

Pricing by arbitrage

Construction Strategy

If Q is such that Z is a Q-martingale we can form the martingale $E_i = E_Q(B_T^{-1}X|\mathcal{F}_i)$. By the theorem we can find a previsible process ϕ so that:

$$E_i = E_0 + \sum_{k=1}^{i} \phi_k \Delta Z_k$$

Pricing by arbitrage

Now let's follow the following strategy: at time i we

-) Buy ϕ_{i+1} units of stock S,
-) Buy $\psi_{i+1} = E_i - \phi_{i+1} B_i^{-1} S_i$ units of cash bond.

At time 0 our portfolio is worth

$$\phi_1 S_0 + \psi_1 B_0 = E_0 = E_Q(B_T^{-1} X).$$

One tick later the portfolio is worth

$$\begin{aligned} \phi_1 S_1 + \psi_1 B_1 &= \phi_1 S_1 + (E_0 - \phi_1 B_0^{-1} S_0) B_1 \\ &= B_1(E_0 + \phi_1(B_1^{-1} S_1 - B_0^{-1} S_0)) \\ &= B_1(E_0 + \phi_1 \Delta Z_1) = B_1 E_1 \end{aligned}$$

Pricing by arbitrage

But, at time 1, according to our strategy we need to hold the portfolio:

$$\phi_2 S_1 + (E_1 - \phi_2 B_1^{-1} S_1) B_1 = B_1 E_1$$

So, our strategy is "self-financing".
At the end, at time T we end up with

$$B_T E_T = B_T E_Q(B_T^{-1} X | \mathcal{F}_T) = B_T B_T^{-1} X = X$$

Now, the price of the claim X must be $E_Q(B_T^{-1} X)$. Any other price will create arbitrage opportunities by following the strategy we have just described.